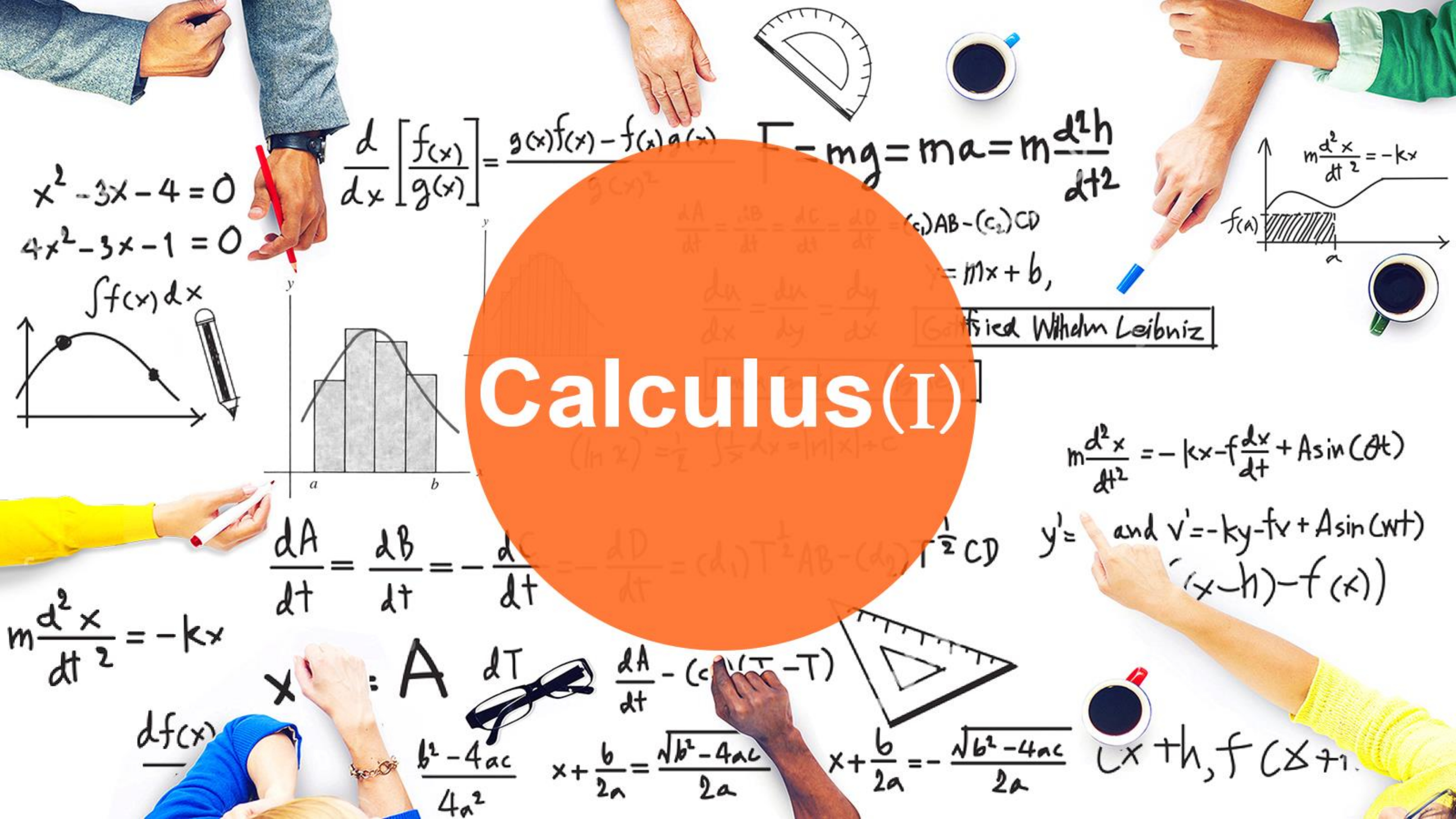
Calculus(Ⅰ)



Supplement for Chapter 3

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Hospital Rule

- $\left[\frac{0}{0}\right],\left[\frac{\infty}{\infty}\right]$
- $\left[0\cdot\infty\right],\left[\infty-\infty\right]$
- $\left[0^{0}\right],\left[1^{\infty}\right],\left[\infty^{0}\right]$
- Summary Question Homework

Definition

If when $x \to a(or\ x \to \infty)$, both $f(x)$ and $F(x) \to 0(or\ \infty)$, then

$$\lim_{\substack{x\to a \\ (x\to\infty)}} \frac{f(x)}{F(x)}$$ is called $\left[\frac{0}{0}\right]$ or $\left[\frac{\infty}{\infty}\right]$.

Eg: $\lim_{x\to 0} \frac{\tan x}{x}$ $\left[\frac{0}{0}\right]$ $\lim_{x\to 0} \frac{\ln \sin ax}{\ln \sin bx}$ $\left[\frac{\infty}{\infty}\right]$

Summary: $\lim_{\substack{x\to a \\ (x\to\infty)}} \frac{f(x)}{F(x)} \Rightarrow \lim_{\substack{x\to a \\ (x\to\infty)}} \frac{f'(x)}{F'(x)}$

Theorem of Hodpital Rule

Theorem A: $\left[\frac{0}{0}\right], \left[\frac{\infty}{\infty}\right]$

If $f(x)$ and $F(x)$ satisfy

(1) $\lim\limits_{x\to a} f(x) = 0(\text{or } \infty)$, $\lim\limits_{x\to a} F(x) = 0(\text{or } \infty)$;

(2) $f(x), F(x)$ is derivative around point a (except a) and $F'(x) \neq 0$;

(3) $\lim\limits_{x\to a} \dfrac{f'(x)}{F'(x)} = A(\text{or } \infty)$;

Then $$\lim_{x\to a} \frac{f(x)}{F(x)} = \lim_{x\to a} \frac{f'(x)}{F'(x)} = A\,(\text{or } \infty).$$

Hospital Rule

1. Find derivatives of molecular and denominator
2. Find the limit and the value is the answer

$$(1)\lim_{x\to a}\frac{f(x)}{F(x)}\left(\frac{0}{0}\right)=\lim_{x\to a}\frac{f'(x)}{F'(x)}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to a}\frac{f''(x)}{F''(x)}\left(\frac{0}{0}\right)=\cdots$$ repeated use of the rule

$(2)\ x\to a+0,\ x\to a-0,$ the rule is applicable

Example 1

Find $\lim\limits_{x\to 0}\dfrac{\cos x-\sqrt{1+x}}{x^3}$. $\left(\dfrac{0}{0}\right)$

$$L=\lim_{x\to 0}\frac{-\sin x-\dfrac{1}{2\sqrt{1+x}}}{3x^2}=\infty.$$

Theorem of Hodpital Rule

Theorem B:

If (1) $\lim_{x\to\infty} f(x) = 0(\text{or } \infty), \lim_{x\to\infty} F(x) = 0(\text{or } \infty)$;

(2) when $|x| > N$, $f(x)$ and $F(x)$ are derivative, and $F'(x) \neq 0$;

(3) $\lim_{x\to\infty} \frac{f'(x)}{F'(x)} = A\,(\text{or } \infty)$;

Then, $\lim_{x\to\infty} \frac{f(x)}{F(x)} = \lim_{x\to\infty} \frac{f'(x)}{F'(x)} = A(\text{or } \infty).$

Proof of Theorem B

Let $x = \frac{1}{z}$, then $x \to \infty$

namely, $z \to 0$, By Theorem A

$$\lim_{x\to\infty}\frac{f(x)}{F(x)} = \lim_{z\to 0}\frac{f\left(\frac{1}{z}\right)}{F\left(\frac{1}{z}\right)} = \lim_{z\to 0}\frac{f'\left(\frac{1}{z}\right)\left(-\frac{1}{z^2}\right)}{F'\left(\frac{1}{z}\right)\left(-\frac{1}{z^2}\right)}$$

$$= \lim_{z\to 0}\frac{f'\left(\frac{1}{z}\right)}{F'\left(\frac{1}{z}\right)} = \lim_{x\to\infty}\frac{f'(x)}{F'(x)} = A$$

Note: Theorem B is applicable when $x \to +\infty(-\infty)$

Example 2

Find $\lim\limits_{x \to +\infty} \dfrac{\dfrac{\pi}{2} - \arctan x}{\sin \dfrac{1}{x}}$. $\left(\dfrac{0}{0}\right)$

$$L = \lim_{x \to +\infty} \frac{\dfrac{-1}{1+x^2}}{\dfrac{-1}{x^2} \cdot \cos \dfrac{1}{x}} = 1$$

NB

(1) Hospital Rule is only applicable to $\left[\frac{0}{0}\right]$ and $\left[\frac{\infty}{\infty}\right]$,

and we can reuse this rule so long as satisfying $\left[\frac{0}{0}\right]$ and $\left[\frac{\infty}{\infty}\right]$;

(2) Notice whether the formula can be simplified;

(3) Every time you use this rule, you should simplify your style;

(4) Hospital Rule is often used in conjunction with other properties of the equivalent infinity and the limit.

Example 3

Find $\lim\limits_{x\to 0}\dfrac{e^x-\sin x-1}{(\arcsin x)^2}$. $\left(\dfrac{0}{0}\right)$

$\because \arcsin x \sim x\ (x\to 0)$

$$L=\lim_{x\to 0}\frac{e^x-\sin x-1}{x^2} \qquad \left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{e^x-\cos x}{2x} \qquad \left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{e^x+\sin x}{2}=\frac{1}{2}.$$

Example 4

Find $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\tan x}{\tan 3x}$. $\left(\dfrac{\infty}{\infty}\right)$

$$L = \lim_{x\to\frac{\pi}{2}} \frac{\sin x \cdot \cos 3x}{\cos x \cdot \sin 3x}$$

$$= -\lim_{x\to\frac{\pi}{2}} \frac{\cos 3x}{\cos x} \quad \left(\frac{0}{0}\right)$$

$$= -\lim_{x\to\frac{\pi}{2}} \frac{-3\sin 3x}{-\sin x} = 3.$$

Example 5

First separate this formula

Find $\lim\limits_{x\to a}\dfrac{\cos x\ln|x-a|}{\ln|e^x-e^a|}$. $\left(\dfrac{\infty}{\infty}\right)$

$$L=\lim_{x\to a}\cos x\cdot\lim_{x\to a}\frac{\ln|x-a|}{\ln|e^x-e^a|}\quad\left(\frac{\infty}{\infty}\right)$$

$$=\cos a\cdot\lim_{x\to a}\frac{\dfrac{1}{x-a}}{\dfrac{e^x}{e^x-e^a}}=\cos a\cdot\lim_{x\to a}\frac{1}{e^x}\cdot\lim_{x\to a}\frac{e^x-e^a}{x-a}$$

use the definition of the derivative of e^x at $x=a$

$$=\frac{\cos a}{e^a}e^a=\cos a.$$

Two aspects of the limitations

(I) When the limit of the derivative ratio does not exist, it can not be concluded that the limit of the function ratio does not exist, then the Hospital Rule can not be used.

Examle 6

(II) We may never get results. It can not be simplified when molecules and denominations are single irrational numbers.

Examle 7

Example 6

Find $\lim_{x\to\infty} \frac{x+\cos x}{x}$ $\left(\frac{\infty}{\infty}\right)$

$$L = \lim_{x\to\infty} \frac{1-\sin x}{1} = \lim_{x\to\infty}(1-\sin x).$$

The limit does not exist

Hospital Rule is invalid.

$$L = \lim_{x\to\infty}\left(1+\frac{1}{x}\cos x\right) = 1.$$

Note: The conditions of use of the Hospital Rule.

Pay attention to NB (3).

Example 7

? Find $\lim\limits_{x\to+\infty}\dfrac{\sqrt{1+x^2}}{x}$ $\left(\dfrac{\infty}{\infty}\right)$

$$L=\lim_{x\to+\infty}\frac{\dfrac{2x}{2\sqrt{1+x^2}}}{1}=\lim_{x\to+\infty}\frac{x}{\sqrt{1+x^2}}\ \left(\frac{\infty}{\infty}\right)$$

$$=\lim_{x\to+\infty}\frac{1}{\dfrac{x}{\sqrt{1+x^2}}}=\lim_{x\to+\infty}\frac{\sqrt{1+x^2}}{x}$$

In fact, $\lim\limits_{x\to+\infty}\dfrac{\sqrt{1+x^2}}{x}=1.$

Example 8

Find $\lim_{x\to+\infty}\frac{\ln x}{x^n}$ (n is positive integer) $\left(\frac{\infty}{\infty}\right)$

$$L=\lim_{x\to+\infty}\frac{\frac{1}{x}}{nx^{n-1}}$$

$$=\lim_{x\to+\infty}\frac{1}{nx^n}=0$$

Note: If n is replaced by $\mu>0$, the limit is still established.

Example 9

?Find $\lim\limits_{x\to+\infty}\dfrac{x^n}{e^{\lambda x}}$ (n is positive integer, $\lambda>0$) $\left(\dfrac{\infty}{\infty}\right)$

$$L=\lim_{x\to+\infty}\frac{nx^{n-1}}{\lambda e^{\lambda x}} \qquad \left(\frac{\infty}{\infty}\right)$$

$$=\lim_{x\to+\infty}\frac{n(n-1)x^{n-2}}{\lambda^2 e^{\lambda x}} \qquad \left(\frac{\infty}{\infty}\right)$$

$$=\cdots\overset{n次}{=}\lim_{x\to+\infty}\frac{n!}{\lambda^n e^{\lambda x}}=0$$

$[0 \cdot \infty]$, $[\infty - \infty]$

Key Change to $\frac{0}{0}, \frac{\infty}{\infty}$.

1. $[0 \cdot \infty]$ Examle 10

Procedure: $0 \cdot \infty \Rightarrow \frac{1}{\infty} \cdot \infty \Rightarrow \frac{\infty}{\infty}$ or $0 \cdot \infty \Rightarrow 0 \cdot \frac{1}{0} \Rightarrow \frac{0}{0}$

2. $[\infty - \infty]$ Examle 11

Procedure: $\infty - \infty \Rightarrow \frac{1}{0} - \frac{1}{0} \Rightarrow \frac{0-0}{0 \cdot 0} \Rightarrow \frac{0}{0}$

Example 10

Find $\lim\limits_{x\to+\infty} x(\frac{\pi}{2}-\arctan x)$. $(\infty\cdot 0)$

$$L=\lim_{x\to+\infty}\frac{\frac{\pi}{2}-\arctan x}{\frac{1}{x}}\quad(\frac{0}{0})$$

$$=\lim_{x\to+\infty}\frac{-\frac{1}{1+x^2}}{-\frac{1}{x^2}}=\lim_{x\to+\infty}\frac{x^2}{1+x^2}=1$$

Example 11

Find $\lim\limits_{x\to 0}(\frac{1}{\sin x}-\frac{1}{x})$. $(\infty-\infty)$

$$L=\lim_{x\to 0}\frac{x-\sin x}{x\cdot\sin x}\quad (\frac{0}{0})$$

$$=\lim_{x\to 0}\frac{1-\cos x}{\sin x+x\cos x}\quad (\frac{0}{0})$$

$$=\lim_{x\to 0}\frac{\sin x}{\cos x+\cos x-x\sin x}$$

$$=0.$$

$[0^0], [1^\infty], [\infty^0]$

Procedure:

$$0^0 \Rightarrow e^{0\cdot\ln 0} \Rightarrow 0\cdot\infty$$

$$1^\infty \Rightarrow e^{\infty\cdot\ln 1} \Rightarrow 0\cdot\infty$$

$$\infty^0 \Rightarrow e^{0\cdot\ln\infty} \Rightarrow 0\cdot\infty$$

Example 12

Find $\lim_{x\to 0^+} x^x$. (0^0)

$$L = \lim_{x\to 0^+} e^{x\ln x} = e^{\lim_{x\to 0^+} x\ln x} \quad (0\cdot\infty)$$

$$= e^{\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}} \left(\frac{\infty}{\infty}\right) = e^{\lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}}$$

$$= e^0 = 1.$$

Example 13

Find $\lim_{x\to 0^+} (\cot x)^{\frac{1}{\ln x}}$ (∞^0)

$$L = \lim_{x\to 0^+} e^{\frac{1}{\ln x}\cdot \ln(\cot x)} = e^{\lim_{x\to 0^+}\frac{\ln(\cot x)}{\ln x}} \quad \left(\frac{\infty}{\infty}\right)$$

$$= e^{\lim_{x\to 0^+}\frac{-\frac{1}{\cot x}\cdot\frac{1}{\sin^2 x}}{\frac{1}{x}}} = e^{-1}.$$

Note: $e^{\frac{\ln(\cot x)}{\ln x}}$, namely, $\exp\left\{\frac{\ln\cot x}{\ln x}\right\}$.

$\exp\{x\}$ is a representation of exponential functions e^{x}

exponent

Example 14

Find $\lim_{x\to\infty}\left(\sin\frac{2}{x}+\cos\frac{1}{x}\right)^{x}$ (1^{∞})

Is there any other way?

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e$$

$$L=\lim_{x\to\infty} e^{x\ln\left(\sin\frac{2}{x}+\cos\frac{1}{x}\right)}$$

$$=e^{\lim_{x\to\infty} x\ln\left(\sin\frac{2}{x}+\cos\frac{1}{x}\right)} \quad (0\cdot\infty)$$

Let $t=\frac{1}{x}$

$$=e^{\lim_{t\to 0}\frac{\ln(\sin 2t+\cos t)}{t}} \quad \left(\frac{0}{0}\right)$$

$$=e^{\lim_{t\to 0}\frac{2\cos 2t-\sin t}{\sin 2t+\cos t}}=e^{2}$$

Example 15

Can not use the Hospital Rule

Find $\lim\limits_{n\to\infty} ne^{-2n}$

The limits of the series

Convert to the limit of the function's infinitive!

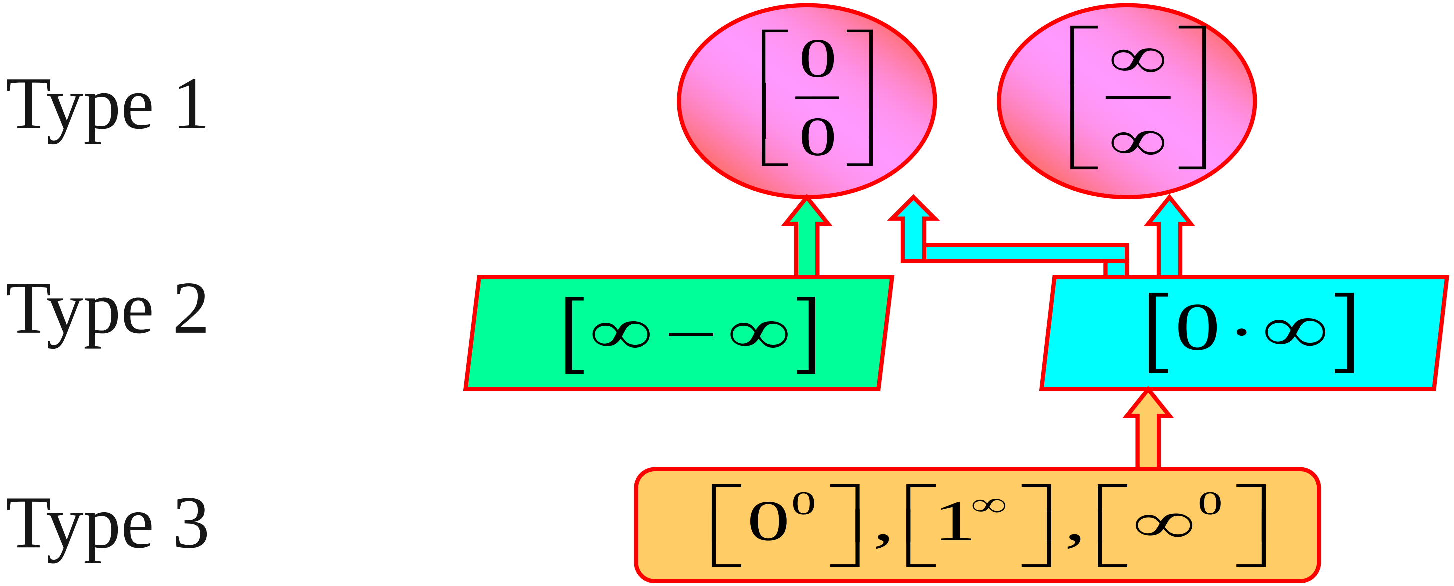
Due to $\lim\limits_{x\to+\infty} xe^{-2x}$ $(0\cdot\infty)$ $= \lim\limits_{x\to+\infty} \dfrac{x}{e^{2x}} \left(\dfrac{\infty}{\infty}\right)$

$$= \lim_{x\to+\infty} \frac{1}{2e^{2x}} = 0$$

and $n\to\infty$ is a special case in $x\to+\infty$

So, $\lim\limits_{n\to\infty} ne^{-2n} = 0$

Summary



(1) If there is a factor of non-zero, the limit can be obtained first.

(2) Any product or business of non-zero infinitesimal factor can be replaced by a simple form of the equivalent of infinitesimal.

Be sure to remember the commonly used infinitely small.

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